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ON THE BEHAVIOR OF DYNAMIC SYSTEMS IN THE VICINITY OF EXISTENCE BOUNDARIES OF PERIODIC MOTIONS

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Violation of stability conditions of existence of multidimensional system periodic motions induced by transition through the discontinuity point of the characteristic of one of the system nonlinear elements is considered. It is shown that at stronger discontinuity a motion close to the disturbed one generally loses its stability, and that the related existence boundary of the stability region becomes dangerous, if no section of a sliding mode makes it appearance. A dangerous boundary always corresponds to a bifurcation associated with the occurence of incompletely elastic collisions. The possibility of unlimited complication of the parameter space structure due to the effect of boundary "blurring" is established. An example of the analysis of that structure is presented, and an estimate is made of the width of the band of boundary blurring.

1. Transition through the discontinuity of a characteristic. Let us consider a piece-wise continuous dynamic system in which the characteristic of one of its nonlinear elements, say $\varphi(x_1)$, has a discontinuity defined as

$$\varphi(x_1) = \begin{cases} \psi(x_1) & (D \leqslant 0) \\ \psi(x_{10}) + (x_1 - x_{10})(\psi'(x_{10}) + q) & (D \geqslant 0) \end{cases} \tag{1.1}$$

when the phase trajectory passes through some surface

$$D(x_1, x_2, \ldots, x_n, t, \mu) = 0 \tag{1.2}$$

where $x_1, x_2, \ldots, t$ are phase coordinates of the system, x_{10} is the coordinate of the discontinuity point, and q and μ are independent parameters (Fig. 1).

We assume that the considered system has a stable periodic solution when $\mu \leqslant 0$, and that a violation of conditions of its existence (C-bifurcation) is defined by the tangency of some section of a phase trajectory lying in the subspace $D < 0$ to surface (1.2) when $\mu = 0$.

We shall investigate the effect of the quantitative index of characteristic q on the system behavior when parameter μ is varied. Thus formulated the problem is that of the system coarseness relative to the nonlinear characteristics of the considered class [1]. The most important aspect of this is the establishment of dangerous boundaries of the region of periodic solution existence whose slightest infraction (by suitable selection of a reasonably small disturbance) results in an uncontrollable increase of deviation of the motion mode from the considered in [2].

We introduce in the analysis the Π point transformation of some fairly smooth surface F with the transformation generated by phase trajectories of the system in the neighborhood of the stable periodic solution trajectory. It is possible to draw through the stationary point M^* which on surface F corresponds to $\mu = 0$, some curve s defined by that the phase trajectories that pass through it are tangent to surface $D = 0$. Such curves divide F into two half-neighborhoods F^+ and F^- which correspond to different equations of the Π^+ and Π^- transformations. We assume that in the M^* neighborhood the transformation is continuous, and that its dependence on phase coordinates and parameters in each of the half-neighborhoods is fairly smooth.

We select the coordinate system on F so that $u_1 = u_2 = \ldots = u_n = 0$ correspond to $\mu = 0$ and the axis u_n normal to boundary s; regions F^+ and F^- then correspond to u_n of different signs. In these coordinates, linearized with respect to $u_1, \ldots, u_n$ and μ, the equations of point transformations are of the following form [3]:

the Π^+ transformation

$$u_k' = \sum_{i=1}^{n} a_{ki}u_i + b_k\mu + \ldots \quad (u_n \geqslant 0) \tag{1.3}$$

and the Π^- transformation

$$u_k' = \sum_{i=1}^{n} a_{ki}u_i + qa_k(q)\,u_n + b_k\mu + \ldots \quad (u_n \leqslant 0) \quad k = 1, 2, \ldots, n \tag{1.4}$$

The form of these equations implies that system (1.3) can be formally obtained from Eqs. (1.4) by setting in these $q = 0$. Hence the characteristic polynomial of transformation (1.4) is of the form

$$\chi(\lambda, q) = \lambda^n + \gamma_1(q)\,\lambda^{n-1} + \ldots + \gamma_{n-1}(q)\,\lambda + \gamma_n(q) \tag{1.5}$$
$$\gamma_i = \alpha_i + q\,\beta_i(q), \quad i = 1, 2, \ldots, n$$

where α_i are coefficients of the characteristic polynomial of transformation (1.3).

On the basis of above assumptions there must exist for $\mu < 0$ the stable stationary point of transformation (1.3)

$$u_k^* = \frac{B_k^+}{\chi(1, 0)}\mu + \ldots, \quad u_n^* = \frac{B_n}{\chi(1, 0)}\mu + \ldots \geqslant 0 \tag{1.6}$$

where B_k^+ and B_n denote related determinants.

The simple periodic solution close to the disturbed one is determined by the stationary point of transformation (1.4)

$$u_k^{**} = \frac{B_k^-}{\chi(1, q)}\mu + \ldots, \quad u_n^{**} = \frac{B_n}{\chi(1, q)}\mu + \ldots \leqslant 0 \tag{1.7}$$

Solution (1.7) exists, when the eigenvalues of the matrix of transformation (1.4) are not equal unity, and it is stable when all roots of $\chi(\lambda, q) = 0$ lie within the unit circle.

It follows from initial assumptions about stability when $q = 0$ and from the continuous dependence $\chi(\lambda, q)$ on the parameter that there exists a fairly small $q^* > 0$ for which all roots lie in the unit circle, and that the condition

$$\chi(1, 0)\ \chi(1, q) > 0, \quad |q| < q^* \tag{1.8}$$

is satisfied.

From formulas (1.6), (1.7), and condition (1.8) we obtain that when parameter μ changes its sign, the coordinate u_n of the stationary point also changes its sign and, consequently, the stable periodic mode (1.6) is transformed into the stable mode (1.7). Thus the C-bifurcation generated by a fairly small discontinuity of the characteristic does not lead to qualitative changes of the system behavior, and is safe.

We shall show that with increasing $|q|$ the periodic solution (1.7) becomes unstable, if, at least one of the $\varepsilon_i = \lim\limits_{|q|\to\infty} \beta_i(q) \neq 0$ and a so-called sliding mode for which the transformation equations differ from (1.4) does not occur on surface $D = 0$.

We use the Schur stability condition and set $\varepsilon_n \neq 0$. Then for $|q| \to \infty$ the first Schur inequality

$$|\gamma_n| = |\alpha_n + q\beta_n(q)| < 1 \tag{1.9}$$

is violated.

Let us now assume that $\varepsilon_n = 0$ and that condition (1.9) holds. In conformity with the procedure for the derivation of Schur inequalities we introduce the polynomial $\chi^*(\lambda, q)$, and form a polynomial of $n-1$ power by the rule

$$\chi_1(\lambda, q) = \lambda^{-1}(\chi - \gamma_n\chi^*) = (1 - \gamma_n^2)\lambda^{n-1} + \ldots + (\gamma_{n-1} - \gamma_1\gamma_n) \tag{1.10}$$
$$\chi^*(\lambda, q) = \gamma_n\lambda^n + \gamma_{n-1}\lambda^{n-1} + \ldots + \gamma_1\lambda + 1$$

The next Schur polynomial is of the form

$$|\alpha_{n-1} - \alpha_1\gamma_n + q(\beta_{n-1} - \beta_1\gamma_n)| < 1 - \gamma_n^2 \tag{1.11}$$

If (1.9) is satisfied and $|q| \to \infty$ the necessary conditions for inequality (1.11) to be satisfied are: $\beta_{n-1}(q) \to 0$ and $\beta_1(q) \to 0$, or

$$\beta_{n-1} - \beta_1\gamma_n \equiv 0 \tag{1.12}$$

Dealing with subsequent Schur inequalities in the same manner, we conclude that on above assumptions they can only be satisfied in the degenerate case of $\beta_n(q) \to 0$, and that the remaining coefficients $\beta_i(q)$ $(i = 1, 2, \ldots, n-1)$ satisfy $n-1$ equations of the kind (1.12). Hence an increase of the discontinuity of the nonlinear characteristic generally results in the loss of stability of the periodic motion that takes place in the presence of the C-bifurcation generated by that discontinuity and is close to disturbed motion. The related boundary of the region of periodic motion existence becomes dangerous.

2. Occurrence of collisions. A particular case of the considered problem, in which $q \to \infty$ and $\varphi(x_1)$ is the elastic characteristic of some joint, corresponds to an absolutely elastic collision at $x_1 = x_{10}$. It is therefore, to be expected that the boundary of periodic motion existence associated with the onset of not completely elastic impacts in any elements of an oscillating structure may prove to be dangerous. This conclusion does not, however, follow directly from Sect. 1, and requires separate investigation.

Let the system phase coordinates be such that x_1 defines the relative displacement of two colliding masses, $x_2 = x_1^{\cdot}$, x_3 defines the displacement of one of these masses, and $x_4 = x_3^{\cdot}$. The reference point is selected so that the impact takes place at $x_1 = 0$ and impact-free motion corresponds to $x_1 < 0$.

We introduce in the analysis the point transformation $M_1 = \Pi(M_0)$ of the half-plane $x_1 = 0$, $x_2 > 0$ generated by the equations of impact-free motion in the neighborhood of the trajectory of the stable initial periodic solution. With this transformation the stationary point $M^* = \Pi(M^*)$ becomes in the case of C-bifurcation $\mu = 0$ the point of contact of that half-plane, while for $\mu > 0$ it corresponds to a motion which, although stable, is unobtainable in a real system, owing to the neglect of the impact interaction at $x_1 = 0$ (Fig. 2, a).

We assume the transformation equations to be of the form

$$\begin{aligned} x_{11} &= 0 = f_1(0, x_{20}, x_{30}, \ldots, x_{n0}, t_0, t_1) \\ x_{i1} &= f_i(0, x_{20}, x_{30}, \ldots, x_{n0}, t_0, t_1) \quad (i = 2, 3, \ldots, n) \end{aligned} \tag{2.1}$$

The characteristic equation obtained in the usual manner from (2.1) is of the form

$$\chi(\lambda) = \begin{vmatrix} \ldots, \dfrac{\partial f_1}{\partial x_{k0}} & , \ldots, \dfrac{\partial f_1}{\partial t_0} + \lambda \dfrac{\partial f_1}{\partial t_1} \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots, \dfrac{\partial f_i}{\partial x_{k0}} - \delta_{ik}\lambda, & \ldots, \dfrac{\partial f_i}{\partial t_0} + \lambda \dfrac{\partial f_i}{\partial t_1} \\ \ldots\ldots\ldots & \ldots\ldots\ldots \end{vmatrix} = \sum_{m=0}^{n} c_m \lambda^{n-m} = 0 \tag{2.2}$$

$$(i, k = 2, 3, \ldots, n)$$

where δ_{ik} is the Kronecker delta and partial derivatives are determined by coordinates $x_{i1} = x_{i0} = x_i^*$, $t_0 = t^*$, and $t_1 = t^* + T$ of the stationary point, and T is the period of motion.

Note the singularity of this polynomial, which is related to the selection of the point transformation. When $\mu \to 0$ the hyperplane $x_1 = 0$ becomes a surface with contact, and the relative velocity $x_2^* = \partial f_1 / \partial t_1 \to 0$. This implies that the coefficient

$$c_0 = (-1)^{2n} x_2^* \to 0 \tag{2.3}$$

In the case of a definitely stable periodic motion, condition (2.3) implies that all other coefficients c_m in Eq. (2.2) also vanish in the presence of a C-bifurcation, which indicates the appearance in the equation of a common factor which also vanishes when $\mu \to 0$.

Let us now consider the near-periodic motion which includes collisions of masses at $x_1 = 0$ (Fig. 2, b). (When a C-bifurcation of periodic motion is induced by the appearance of additional collisions, the occurence of a particular sliding mode is possible in certain cases [4]). If the impact is considered to be instantaneous with the velocity recovery coefficient R, we have the known relationship between the postand pre-collision velocities, respectively x_{20}^+ and x_{40}^+, and x_{20} and x_{40}, of the form

$$x_{20}^+ = -Rx_{20}, \quad x_{40}^+ = x_{40} + \rho x_{20}, \quad \rho = \frac{1+R}{1+\varkappa} \tag{2.4}$$

where $\varkappa$ denotes the ratio of colliding masses. The stationary transformation point

$$\begin{aligned} 0 &= f_1\,(0, -Rx_{20}, x_{30}, x_{40} + \rho x_{20}, \ldots, x_{n0}, t_0, t_1) \\ x_{i1} &= f_i\,(0, -Rx_{20}, x_{30}, x_{40} + \rho x_{20}, \ldots, x_{n0}, t_0, t_1) \\ (i &= 2, 3, \ldots, n) \end{aligned} \tag{2.5}$$

corresponds to a periodic solution. The characteristic matrix of Eqs. (2.5) formally differs from (2.2) only by the elements of the first column

$$-R\frac{\partial f_i}{\partial x_{20}} - \delta_{i2}\lambda + \rho\frac{\partial f_i}{\partial x_{40}} \quad (i = 1, 2, \ldots, n) \tag{2.6}$$

Since at the C-bifurcation the considered periodic motions merge, the partial derivatives in both characteristic equations are the same. From this and directly from (2.2) and (2.6) we obtain that for the characteristic polynomial

$$\chi(\lambda, R) = \sum_{m=0}^{n} C_m(R)\,\lambda^{n-m}$$

of transformation (2.5) $C_0(R) = c_0 \to 0$ and $C_n(R) = -Rc_n \to 0$ when $\mu \to 0$. However, if not all $\partial f_i / \partial x_{20} \to 0$, at least one of the remaining coefficients $C_m(R)$ $(m = 1, \ldots, n-1)$ does not vanish for any $R \in (0,1)$. But then at least one of the stability conditions for motions generated at the C-bifurcation with additional collisions is violated.

Note that this implies the instability of periodic motions with collisions which are close to harmonic (i.e. collision-free motions). This indicates the necessity of a stricter proof of the validity of the method of harmonic linearization for analyzing periodic oscillations of systems with collisions.

3. Structure of the neighborhood and the effect of existence boundary blurring. The detection in the existence boundary of sections with loss of stability poses the problem of determination of the kind of motion which is established at transition through such sections and of the properties of such motions. For defining the C-bifurcation we shall use the criteria derived in [3]. If the characteristic polynomial $\chi(\lambda, q, \mu)$ of the considered transformation is known, the condition of safe transition involving the change of sign of parameter μ is of the form

$$\chi(1, 0, 0)\,\chi(1, q, 0) > 0, \quad |q| < q_+, \quad \chi(1, q_+, 0) = 0 \tag{3.1}$$

The unsafe transition in which the initially stable motion $(q = 0)$ merges with the unstable motion $(q \neq 0)$ and then vanishes is defined by

$$\chi(1, 0, 0)\,\chi(1, q, 0) < 0, \quad |q| > q_+ \tag{3.2}$$

The condition of generation of a motion of doubled period whose phase trajectory does not reach the discontinuity surface $D = 0$ (1.2) in a single revolution and in two revolutions penetrates the subspace $D > 0$ is

$$\chi(-1, 0, 0)\,\chi(-1, q, 0) < 0, \quad |q| > q_-, \quad \chi(-1, q_-, 0) = 0 \qquad (3.3)$$

Note that condition (3.3) defines two alternatives: a safe one when $|q| < q_{++}$ and an unsafe one when $|q| > q_{++}$, which are similar to cases (3.1) and (3.2). The parameter q_{++} corresponds here to the loss of stability of a two-revolution motion.

By supplementing the above conditions with the known properties of the system near the stability region boundary, namely, that vanishing of the boundary $\chi(1, q, \mu) = 0$ of the Jacobian of transformation corresponds to the merging of stable and unstable motions of the same kind and that the boundary $\chi(-1, q, \mu) = 0$ corresponds to the generation of two-revolution motion, we obtain the following simplest qualitative patterns of the structure of the neighborhood of the considered C-boundary $\mu = 0$ in the plane of the two parameters μ and q (Fig.3). This kind of motion is conventionally represented there by a section of the phase trajectory, the shaded area corresponds to unstable motions, the presence of discontinuities relates to the penetration of the motion phase trajectory into region $D > 0$, and the two adjacent sections correspond to a two-revolution periodic motion.

The shaded area in Fig.3b corresponds to a C-bifurcation transition between two two-revolution modes with one and two-phase trajectory penetrations into the subspace $D > 0$. The above analysis is fully applicable to the structure of the indicated area which with increasing $|q|$ may not only become complex in itself but, also, generate new bifurcation nodes which become origins of new C- and motion stability boundaries that are even more complex than the initial ones (see (1.6) and (1.7)).

If simultaneously the whole region of solution ambiguity in the neighborhood of the C-boundary $\mu = 0$ remains fairly narrow, it is that region as a whole, not the numerous individual boundaries of transition and stability contained in it, that is of practical interest. It is then correct to speak of blurring of the existence boundary of periodic motion (1.6).

With increasing structure complexity an exact analysis of properties of particular kinds of motions in the boundary blurring area loses practical meaning. Depending on the object of investigation, it may prove to be advantageous either to use the statistical approach to the study of such motions or to modify the considered model so as to eliminate the generation of stochastic processes.

Width of the band of blurring of the existence boundary can be estimated by investigating the behavior of some of the stability boundaries. For the parameter space region $\mu < 0$ it is the stability boundary of a generally manifold mode that corresponds to one of the roots of the characteristic equation being equal $+1$. For region $\mu > 0$ it is the stability boundary of periodic motion (1.7) that corresponds to the conversion of one of the roots to -1 (Fig.3).

4. Example. Let us investigate the width and the structure of the blurring band at one of the sections where there exists a linear system of forced oscillation mode disturbed by a fixed motion limiter. The dimensionless equations of motion of such system are of the form

$$\begin{aligned} &x^{\cdot\cdot} + 2cx^{\cdot} + x = \cos\omega\tau \qquad (x < d) \\ &x_+^{\cdot} = -Rx_-^{\cdot} \qquad (x = d) \end{aligned} \qquad (4.1)$$

where c is the coefficient of linear friction and R is the coefficient of velocity recovery at impact.

We denote by $\Gamma(n, k)$ the stable motion with k impacts in the course of one period equal $2\pi n / \omega$. Stable forced linear oscillations $\Gamma(1, 0)$ obviously exist in system (4.1) when $\mu = d/d_* - 1 > 0$, where d_* is the amplitude of such oscillations. Analysis shows that at the boundary of $\mu = 0$ a simple safe transition to any stable motion mode of the kind $\Gamma(n, 1)$ is impossible. Note that on the indicated boundary not only individual $\Gamma(n, k)$ but complete sets of such motions may be generated [5]. However, the stated problem can be solved by restricting the analysis to a set of motions of the kind $\Gamma(n, 1)$.

When $c \ll 1$ the equation for the stability boundary $\Gamma(n, 1)$ lying in region $\mu > 0$ and corresponding to one of the roots equal $+1$ may be written as [5]

$$\mu(2+\mu) - \omega^2\left(\frac{1-R}{1+R}\operatorname{tg}\frac{\theta}{2} - c\left(1 - \frac{\theta}{\sin\theta}\right)\right)^{-2} = 0, \quad \theta = \frac{2\pi n}{\omega}, \quad \sin\theta < 0 \qquad (4.2)$$

Within the existence region of any of the periodic solution (4.1) the dependence of this solution on parameters is continuous. Hence, the wider the region of ambiguity of solutions comprised between curve (4.2) and the C-boundary of $\mu = 0$, the more "dangerous" the collapse of forced linear oscillations at that boundary may prove to be. It is known that similar collapses of systems whose elastic properties are somewhat stiff occur in the region of frequencies higher than resonance.

Let us now consider region of frequencies $\omega < 1$ lower than resonance. Stable modes $\Gamma(n, 1)$ for $\mu > 0$ occur in conformity with (4.2) in the frequency intervals

$$1 + \frac{1}{2n} < \frac{1}{\omega} < 1 + \frac{1}{n} \qquad (4.3)$$

The width of each of these regions estimated over mean values of ω in intervals (4.3) tends to zero with increasing n when $\omega \to 1$ in accordance with the approximate formula

$$\mu \approx {}^1/_2\omega^2\left(\frac{1-R}{1+R} + c\,\frac{{}^3/_2\pi}{1-\omega}\right)^{-2} \qquad (4.4)$$

From the side of $\mu > 0$ region S, whose structure becomes more complex in the interval ${}^2/_3 < \omega < 1$, extends to the stability boundary of the single impact mode $\Gamma(1, 1)$, which corresponds to the equality $\lambda = -1$ for one of the roots of the characteristic polynomial.

In the case of $1 - \omega \ll 1$ and energy dissipation produced only by not entirely elastic collisions the equation of that boundary may be presented as

$$\mu \approx -\left(\frac{1}{\omega} - 1\right)\frac{(1+R)^2}{1+R^2} \qquad (4.5)$$

Thus in the neighborhood of $\omega = 1$ region S lies between the boundaries (4.4) and (4.5) which originate at point $\omega = 1$, $\mu = 0$. The size of S increases somewhat with increasing R. For $R = 0.6$ that region is shown in Fig. 4. The number m of various modes of the kind $\Gamma(n, 1)$ that may occur at one and the same frequency is determined in conformity with (4.3) by the inequality

$$1/(2n) < 1/\omega - 1 < 1/(n+m)$$

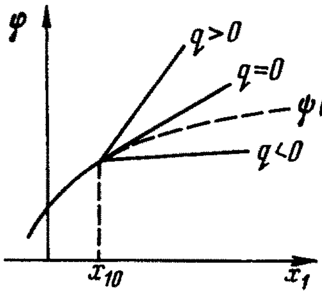

Fig. 1

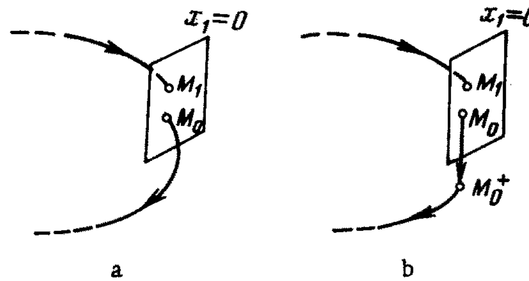

Fig. 2

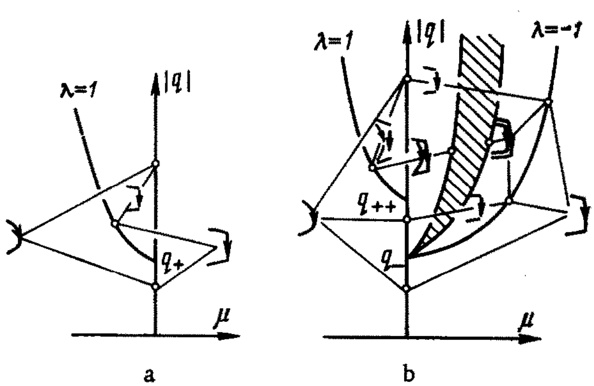

Fig. 3

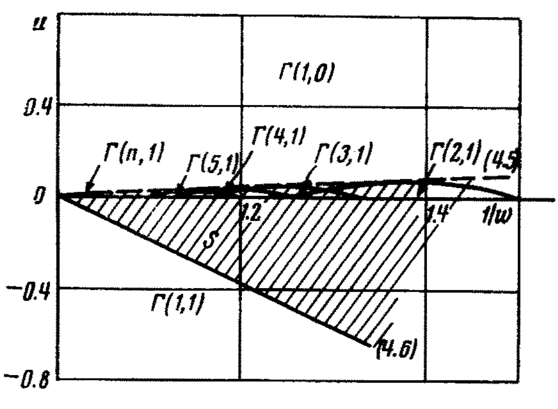

Fig. 4

and increases indefinitely with the approach to resonance frequency. The variation range of parameter μ within S then tends to zero. The increasing complexity of that structure makes unattractive the investigation of individual motions in S. This does not, however, imply the unavoidability of statistical analysis. If maximum deflections of an oscillating system and not the phases of separate collisions are of interest, it is advisable to use a different mathematical model for that part of the S region where the effect of blurring of the existence boundary of $\Gamma(1, 0)$ is strongest. The simplest variant of this can be a linear model in which energy dissipation owing to not entirely elastic collisions is taken into account. The related input to the "equivalent" linear friction coefficient c° can be estimated by formula

$$c^\circ = c + \frac{(1-\omega)(1-R)}{{}^3/_2\pi(1+R)} \tag{4.6}$$

which was readily obtained in formula (4.4) as a supplement to coefficient c.

This makes it possible to investigate individual motions for fairly large n, since it is possible to avoid stochasticity by substituting a linear model with the friction coefficient (4.6) and $\omega > 2n/(1+2n)$ for the original model (4.1).

We note in conclusion that the results of the present investigation are in good agreement with those carried out on an analog computer for the fine structure of motions in system (4.1) [6]. In fact, in the subresonance region the kinds of motions were becoming more complex the closer the parameters were selected to the boundary $\mu = 0$ and frequency $\omega = 1$. Appearance of stochasticity is to be expected in cases when existence regions of individual parameter become so narrow that they are spanned over by fluctuations of the analog computer elements. The motions called in [6] quasi-periodic apparently correspond to such cases.

The obtained theoretical results make it possible to better define the disposition of existence boundaries denoted in [6] by ρ_1. Since they are unsafe, they must shift into the region of solution ambiguity.

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